Is This Construction Legal?

To construct μ from ρ, we may have to perform an infinite number of transformations. Since we can't actually do that, we need to define μ more carefully and show that it is a behavior of $\mathcal{F}$. The proof is not hard, but you may find it confusing if you have not studied infinite sequences in a math course.

We say that an infinite sequence $\mu_0, \mu_1, \ldots$ of behaviors *converges* iff it satisfies the following condition: For every postive integer p, there exists a positive integer q such that every μ_i with $i > q$ has the same prefix of length p. If $\mu_0, \mu_1, \ldots$ converges, then there is a unique behavior μ such that, for every positive integer p, there is a q such that the p^{th} state of μ is the p^{th} state of μ_i for all $i > q$. In that case, we call μ the *limit* of the sequence $\mu_0, \mu_1, \ldots$.

We now inductively construct a (finite or infinite) sequence $\mu_0, \mu_1, \ldots$ of behaviors as follows. We start with $\mu_0 = \rho$. For every $j > 0$, find the j^{th} occurrence of an r_1 step in μ_{j-1}. If there is none, or if there is no later r_n step, the sequence stops with μ_{j-1}. Otherwise, construct μ_j from μ_{j-1} by moving r_i steps right (for $i < k$) or left (for $i > k$) so that the j^{th} r_1 step is immediately followed by $r_2, \ldots, r_n$ steps. This produces a finite or infinite sequence of behaviors μ_i that all satisfy $\mathcal{F}$. There are three cases:

1. The sequence of $\mu_0, \mu_1, \ldots$ is infinite. In this case, the sequence converges and we define μ to be its limit.

2. The sequence is finite and ends with μ_p, and the last r_i step of μ_p has $i = n$. In that case, we let μ equal μ_p.

3. The sequence is finite and ends with μ_p, and the last r_i step of μ_p has $i < k$. In this case, we define an infinite sequence $\nu_1, \nu_2, \ldots$ of behaviors as follows. Suppose that the last r_i step is the q^{th} step of μ_p. Define ν_j to be the behavior obtained from μ_p by moving the last $r_1, r_2, \ldots, r_i$ steps to the right past the $q+j^{\text{th}}$ step. Each ν_j is a behavior of $\mathcal{F}$, and the sequence $\nu_1, \nu_2, \ldots$ converges. Let μ be its limit.

In each case, μ either satisfies $\mathcal{F}$ (case 2) or is the limit of a sequence of behaviors, each of which satisfies $\mathcal{F}$ (cases 1 and 3). The following argument shows that if a sequence $\tau_1, \tau_2, \ldots$ converges to the limit μ and each τ_i is a behavior of $\mathcal{F}$, then μ is a behavior of $\mathcal{F}$. The proof is by contradiction. Suppose μ is not a behavior of $\mathcal{F}$. Since $\mathcal{F}$ has the form $\mathit{Init} \wedge \Box[\mathit{Next}]_{\mathit{vars}}$, either the initial state of μ does not satisfy *Init* or some step of μ is not a *Next* step. By definition of limit, that initial state or step must be in infinitely many of the τ_i. This is impossible, since all the τ_i are behaviors of $\mathcal{F}$, so μ is also a behavior of $\mathcal{F}$.

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