Condorcet2: $c \succeq^+ d$ holds, for any elements c and d of $\mathcal{CS}$.

$\langle 1 \rangle 1$. Let c be an arbitrary element of $\mathcal{CS}$ and define

$$D \;\triangleq\; \{d \in \mathcal{CS} : \neg(c \succeq^+ d)\}$$

It suffices to assume D is nonempty and obtain a contradiction.

Proof: Obvious.

$\langle 1 \rangle 2$. $d \succ c$ holds, for all $d \in D$.

Proof: $\neg(c \succeq^+ d)$ implies $\neg(c \succeq d)$ (because $c \succeq d$ implies $c \succeq^+ d$ by definition of the transitive closure), and $\neg(c \succeq d)$ equals $d \succ c$ by definition of $\succeq$.

$\langle 1 \rangle 3$. For all $d \in D$ and all $e \in \mathcal{CS} \setminus D : d \succ e$.

$\langle 2 \rangle 1$. It suffices to assume $d \in D$, $e \in \mathcal{CS} \setminus D$, and $\neg(d \succ e)$ and obtain a contradiction

Proof: Obvious.

$\langle 2 \rangle 2$. $c \succeq^+ e$

Proof: By the assumption of 3.1, which implies $e \notin D$, and the definition of D.

$\langle 2 \rangle 3$. $e \succeq d$

Proof: By the assumption of 3.1, which asserts $\neg(d \succ e)$, and the definition of *succeq*.

$\langle 2 \rangle 4$. Q.E.D.

Steps 3.2 and 3.3 and the definition of the transitive closure imply $c \succeq^+ d$, which by the definition of D contradicts the step 3.1 assumption $d \in D$.

$\langle 1 \rangle 4$. D is a dominating set.

Proof: We must prove that if $d \in D$, then $d \succ e$ for any candidate e not in D. If e is not in $\mathcal{CS}$, this follows because $\mathcal{CS}$ is a dominating set. If $e \in \mathcal{CS}$, then $e \in \mathcal{CS} \setminus D$ and this follows from step 3.

$\langle 1 \rangle 5$. Q.E.D.

Proof: D is a subset of $\mathcal{CS}$ by definition. It is a proper subset of $\mathcal{CS}$ because $c \in \mathcal{CS}$ by the step 1 assumption and step 2 implies $c \notin D$. Therefore, step 4 implies that $\mathcal{CS}$ is not the smallest dominating set, which is a contradiction.

CLOSE